Proving Theorems in Łukasiewicz's Simple Propositional Calculus

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Introduction and Comments

(0.0) We take our *propositional calculus* (alternatively called a *sentential logic*, *propositional logic*, etc) to be called L_1.

(0.1) We allow only one rule of inference *Modus Ponens* (MP):

(MP) $A, A \rightarrow B \vdash B$

(0.2) For our purposes, *Modus Ponens* will follow two lines in a proof.

Example:	1]	$A \rightarrow (A \rightarrow A)$	S1; Sub(B, A)
	2]	$(A \rightarrow (A \rightarrow A)) \rightarrow ((A \rightarrow A) \rightarrow (A \rightarrow A))$	AS2; Sub(B, A); Sub(C, A)
	3]	$((A \rightarrow A) \rightarrow (A \rightarrow A))$	MP 1], 2]

(0.3) We allow only two logical operators (logical connectives):

$\{\rightarrow\}$
$\{\neg\}$

(0.4) The *well-formed formulae* (wff) of L_1 are recursively defined as follows:

(i) Any δ, where δ is a propositional variable of L_1, is a formula.
(ii) If δ is a formula then, $\neg\delta$ is a formula.
(iii) If δ and φ are formulas then, $\delta \rightarrow \varphi$ is a formula.
(iv) Any combination of the above is a wff.
(v) There are no other wff.

(0.5) We allow only three *axiom schemata*:

(AS1) $A \rightarrow (B \rightarrow A)$
(AS2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
(AS3) $(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$

(0.6) An *axiom schemata* differs from an *axiom*, in that the former lays down any wff of a particular form as being an *axiom* of the logic, whereas the latter takes a specific wff as a fundamental truth. *Axioms* are used to derive other statements within the logic, using the rules of inference. As a note, a *schema* is a general meta-level rule that describes the behavior of *axioms* within a logic (Axiom of Replacement in ZFC Set Theory). *Schema* and *scheme* are used interchangeably for the singular. *Schemas*, *schemata*, *schemes* are used interchangeably for the plural.

(0.7) Thus, any wff of L_1 can be *substituted uniformly* through one of the *axiom schemata* (AS1), (AS2) and (AS3).

Examples:	(i)	$A \rightarrow (A \rightarrow A)$	AS1; Sub(B, A)
	(ii)	$B \rightarrow (A \rightarrow B)$	AS1; Sub(B, A); Sub(A, B)

(0.8) We define a *theorem* to be a statement proved from the application of our inference rules and *axiom schemata* alone, that is to say without any additional *premises* (assumptions). All *theorems* of L_1 are *axioms* given that they follow from the *schemata* with inference rules.

(0.9) For our present investigation we will prove some of the *theorems* of L_1.

(0.10) Outermost parenthesis can be dropped.

(0.11) We will use [ ] and { } with the usual parenthetical notation ( ) as a means to avoid confusion and to assist in dropping outermost, or unnecessary, parenthesis.

(0.12) When a *premise* is made we will denote it P#. Proofs derived using *premises* will be called *rules*.

(0.13) A *rule* can be used on an acceptable *premise*, or on a derived statement.

Proving Theorems and Rules

[**Th1**]	$\vdash (A \to A)$	
1]	$A \to ((A \to A) \to A)$	AS1; Sub(B, $[A \to A]$)
2]	$[A \to \{(A \to A) \to A\}] \to [\{A \to (A \to A)\} \to (A \to A)]$	AS2; Sub(C, A); Sub(B, $[A \to A]$)
3]	$\{A \to (A \to A)\} \to (A \to A)$	MP 1], 2]
4]	$A \to (A \to A)$	AS1; Sub(B, A)
5]	$(A \to A)$	MP 3], 4]
QED		

[**R1**]	$(A \to B), (B \to C) \vdash (A \to C)$	
1]	$(A \to B)$	P1
2]	$(B \to C)$	P2
3]	$(B \to C) \to [A \to (B \to C)]$	AS1; Sub(A, $[B \to C]$); Sub(B, A)
4]	$A \to (B \to C)$	MP 2], 3]
5]	$[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$	AS2
6]	$(A \to B) \to (A \to C)$	MP 4], 5]
7]	$(A \to C)$	MP 1], 6]
QED		

[R2]	$\neg\neg A \vdash A$	
1]	$\neg\neg A$	P1
2]	$\neg\neg A \to (\neg\neg\neg\neg A \to \neg\neg A)$	AS1; Sub(A, $\neg\neg A$); Sub(B, $\neg\neg\neg\neg A$)
3]	$(\neg\neg\neg\neg A \to \neg\neg A)$	MP 1], 2]
4]	$(\neg\neg\neg\neg A \to \neg\neg A) \to (\neg A \to \neg\neg\neg A)$	AS3; Sub(A, $\neg\neg\neg A$); Sub(B, $\neg A$)
5]	$(\neg A \to \neg\neg\neg A)$	MP 3], 4]
6]	$(\neg A \to \neg\neg\neg A) \to (\neg\neg A \to A)$	AS3; Sub(B, $\neg\neg A$)
7]	$(\neg\neg A \to A)$	MP 5], 6]
8]	A	MP 1], 7]
QED		